Nonlinear Quantum Dynamics

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Nonlinear dynamics and dynamical chaos in classical systems is a familiar everyday occurrence. However, attempts to find chaos in the quantum Schrödinger equation for the wave function, or, more generally, the quantum Liouville equation for the density matrix, have all failed. This is due not only to the linearity of the equations but also due to the Hilbert space structure of quantum mechanics which, via the uncertainty principle, does not allow the formation of fine-scale structure in phase space, thus precluding chaos in the sense of classical trajectories. Indeed, some people have even wondered if this behavior constitutes a fundamental failure of quantum mechanics in describing the real world.

A deeper look at this issue reveals that the question needs to be posed more carefully. Quantum mechanics is intrinsically probabilistic, thus the correct formal analogy to the quantum Liouville equation is the corresponding classical Liouville equation describing the evolution of the classical phase space distribution function. If, for isolated systems, we decide that expectation values—averages over the classical or quantum distributions—are the only relevant dynamical variables, then it is easy to show that there is no chaos in the quantum and classical Liouville equations for smooth distributions! This directly follows from unitarity in the quantum case and symplectic evolution in the classical case [1]. The situation, both classically and quantum mechanically is thus very much the same.

However, the evolution of an expectation value of an isolated system does not represent the evolution of an observed system, as in a real experiment. This is equally true classically or quantum mechanically. The correct description of realistic systems must take into account the interaction with the experimental probe and include the changes in the evolution of the distribution function as a consequence of the information extracted during the observation process. This formalism has been developed over the last 30 years, primarily by quantum opticians and mathematicians and is now being applied in quantum control theory and in wider contexts, such as measurements on condensed matter systems.

The basic dynamical equation for observed systems is termed the conditional Liouville equation. This equation is nonlinear due to the term that incorporates the change in the estimation of the system state given the observational record. In the quantum case, there is also a diffusive term that describes the unavoidable quantum backaction noise of the measurement. Previously, we had

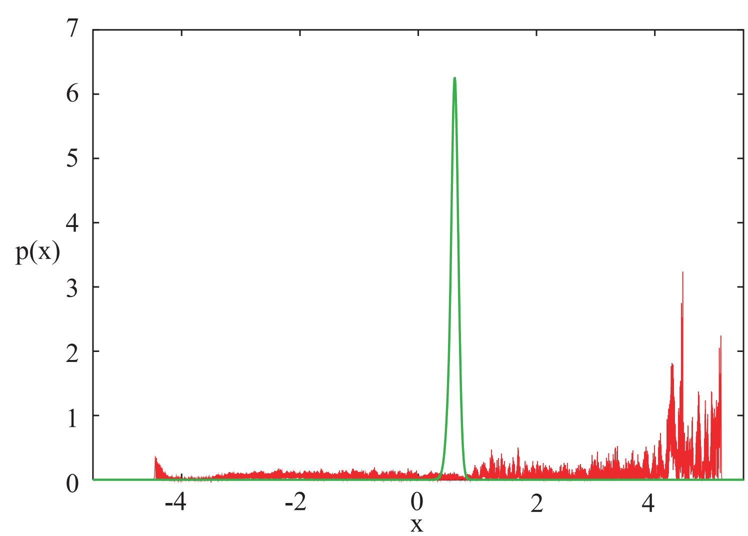

Figure 1—
The position distribution for an observed system (green) compared to the position distribution for the unobserved system (red) from a high-resolution numerical investigation. Note the sharp localization of the observed distribution.

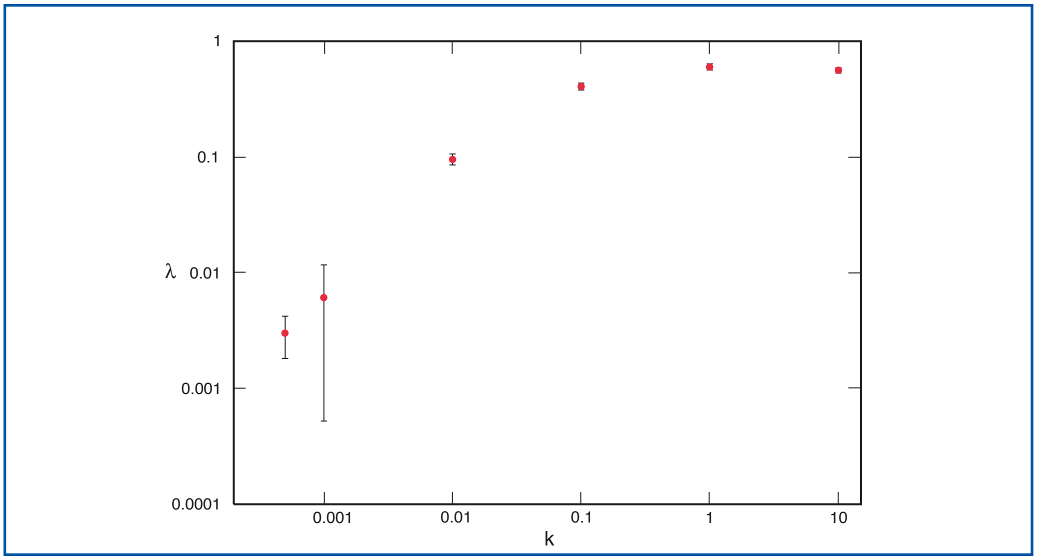

Figure 2—
The quantum Lyapunov exponent for the Duffing oscillator at a fixed value of the Planck constant, but with varying strength of measurement, k. As k increases, the exponent converges to the classical value. At small values of k, the evolution is not localized as sharply as in Fig. 1, yet the Lyapunov exponent is finite.

established the conditions under which the quantum evolution was consistent with classical trajectory results, and hence, consistent with classical chaos [2]. Basically this happens when the observation localizes the state sufficiently so that it can be thought of as "sharp" in a trajectory sense, yet the backaction noise is small enough that the trajectory is not noisy.

But what happens when the quantum evolution is not localized and far from the classical limit? Can there still be chaos in such (observed) quantum systems? We have recently succeeded in defining and computing Lyapunov exponents (the rigorous quantifiers of chaos) for such systems. We have been able to show that intrinsic quantum chaos—in the sense just outlined—exists, not only for weakly measured quantum systems which satisfy the classical limit as the measurement strength is increased (hence the state is more localized), but also when the Planck constant is large enough on the scale of the accessible phase space that no classical limit can exist [1]. The new results have been made possible largely due to parallel supercomputer calculations carried out under the Los Alamos National Laboratory Institutional Computing Initiative.

Interestingly enough, there is a classical analog to the quantum behavior just described. If a classical system is driven by environmental noise, and the noise is sufficiently strong, the classical state resulting from the solution of the conditioned Liouville equation is also not localized in phase space and can have a Lyapunov exponent which is nonzero but different from the classical trajectory Lyapunov exponent. Since the nonlocalized distributions generated by nonlinear Hamiltonians can be quite different depending on whether classical or quantum calculations are performed [3], the Lyapunov exponents in the two cases are also expected to be different. This issue is now under investigation.

Finally, the experimental state of the art is advancing at a rapid pace. It is expected that in a few years experiments in cavity quantum electrodynamics (QED) and nanomechanics will be able to provide controlled testing grounds for the "real" quantum chaos.

[1] S. Habib, K. Jacobs, and K. Shizume, quant-ph/0412159.
[2] T. Bhattacharya, S. Habib, and K. Jacobs, *Phys. Rev. Lett.* **85**, 4852 (2000); *Phys. Rev. A* **67**, 042103 (2003); S. Ghose et al., *Phys. Rev. A* **69**, 052116 (2004).
[3] S. Habib, K. Jacobs, H. Mabuchi, R. Ryne, K. Shizume, and B. Sundaram, *Phys. Rev. Lett.* **88**, 040402 (2002).